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Differentiability properties for a class of non-convex functions

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Abstract Closed sets $K \subset \mathbb{R}^n$ satisfying an external sphere condition with uniform radius (called φ -convexity or proximal smoothness) are considered. It is shown that for \mathcal{H}^{n-1} -a.e. $x \in \partial K$ the proximal normal cone to K at x has dimension one. Moreover if K is the closure of an open set satisfying a (sharp) nondegeneracy condition, then the De Giorgi reduced boundary is equivalent to ∂K and the unit proximal normal equals \mathcal{H}^{n-1} -a.e. the (De Giorgi) external normal. Then lower semicontinuous functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with φ -convex epigraph are shown, among other results, to be locally BV and twice \mathcal{L}^n -a.e. differentiable; furthermore, the lower dimensional rectifiability of the singular set where f is not differentiable is studied. Finally we show that for \mathcal{L}^n -a.e. x there exists $\delta(x) > 0$ such that f is semiconvex on $B(x, \delta(x))$. We remark that such functions are neither convex nor locally Lipschitz, in general. Methods of nonsmooth analysis and of geometric measure theory are used.

1 Introduction

In optimal control or in the theory of viscosity solutions of partial differential equations, semiconcave functions play an important role (see, e.g., the monographs, [5] and [9]). As an example, we mention the fact that some classes of PDE's admit a unique semiconcave solution (see [5, Chapter II]), or that, under suitable controllability assumptions, the time optimal function is shown to

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be semiconcave (or semiconvex, see [9, Chapter 8] and references therein). Semiconcavity, together with the dual concept of semiconvexity, is also considered a good regularity property, in between Lipschitz continuity and \mathcal{C}^1 -regularity (see, again, [5] and [9]). For example, the Euclidean distance from a closed set is semiconcave, and this is in a sense an optimal result, since in general this function is not smooth.

Semiconvex functions are, essentially, quadratic perturbations of convex functions. Therefore, though being not necessarily convex, they inherit from convexity some regularity properties, such as local Lipschitzianity and a.e. double differentiability in the interior of their domain. Moreover, their epigraph may admit corners, as such functions are not necessarily smooth, but those corners may occur only downwards (we recall that this last property is usually called (Clarke) regularity in nonsmooth analysis).

Aim of this paper is studying a class of functions which enjoy Clarke regularity, but are not necessarily locally Lipschitz continuous, yet being, among other things, a.e. twice differentiable. The simplest example illustrating our work is $f(x) = \sqrt{|x|}$, but less trivial functions indeed belong to this class, such as the minimum time to reach the origin for the double integrator (rocket railroad car model, see No. 14 in the Examples of φ -convex functions in Sect. 3.1 below). This last example is in a sense the motivating point of our analysis. In fact, it suggests that the functions which are studied in the present paper may be a reasonable candidate as a regularity paradigm for some optimal control problems, and therefore for solutions of some partial differential equations. Actually, in [14] a class of minimum time functions is shown to belong exactly to the class analyzed in the present work. The key point which identifies the functions studied in this paper is the fact that their epigraph satisfies a kind of external sphere condition, with (locally) uniform radius. Actually, semiconvex functions are identified – within the class of locally Lipschitz functions – by exactly this requirement on their epigraph. By dropping the local Lipschitzianity we therefore make a generalization which seems to be natural. Sets with this property were deeply studied as generalizations of convex sets, mainly in connection with uniqueness of the metric projection and with smoothness of the distance function, both in finite [22] and in infinite dimensions (see, e.g., [13, 26]). Numerous equivalent definitions of this property were given independently by various authors. Among them, we choose the denomination “ φ -convexity,” as it better emphasizes the connections with convexity that we want to analyze. φ -convex sets are known to enjoy, in a neighborhood, some properties that convex sets satisfy globally, the reason being the radius of the external sphere, which is locally bounded away from zero (and continuous) for φ -convex sets, while it is arbitrarily large for convex sets (see Sect. 3.1 below).

In Sect. 4 we prove some regularity properties of φ -convex sets, which enlarge the range of analogies between φ -convex and convex sets. We show that a φ -convex set K admits \mathcal{H}^{n-1} -a.e. on its boundary a unique unit (proximal) normal vector. Moreover, if K is the closure of an open set satisfying a kind of nondegeneracy condition, we show that the reduced boundary (in the sense of De Giorgi) coincides \mathcal{H}^{n-1} -a.e. with the topological boundary, and the De Giorgi external normal coincides with the proximal unit normal. The sharpness of the nondegeneracy assumption is shown through an example.

Then we study the main object of our analysis, lower semicontinuous functions with φ -convex epigraph. First, we compare this property with the φ -convexity of

functions (see [23]), which was introduced in connection with evolution equations driven by nonconvex functionals. We show that φ -convexity of the epigraph is a particular case of φ -convexity of functions, and provide examples of φ -convex functions without a φ -convex epigraph. The main properties of functions with φ -convex epigraph are studied in Sects. 5, 6 and 7. We show that a function f satisfying this assumption has the following properties:

- (i) f is \mathcal{L}^n -a.e. (strictly) differentiable;
- (ii) for \mathcal{L}^n -a.e. x , there exists $\delta = \delta(x) > 0$ such that $f|_{B(x,\delta)}$ is Lipschitz continuous and semiconvex;
- (iii) f is \mathcal{L}^n -a.e. twice differentiable.

Moreover, such functions are BV_{loc} in the interior of their domain, but their differential is not necessarily BV_{loc} ; moreover, they do not belong necessarily to Sobolev spaces like $W_{\text{loc}}^{1,\infty}$ or $W_{\text{loc}}^{2,1}$. Finally, we study the set Σ where f is not differentiable, showing that Σ may be written as the union of Σ_∞ , the set where f is not subdifferentiable, and Σ_k , the sets where the dimension of the (proximal) subdifferential of f is at least k ($k = 1, \dots, n$), and Σ_k is countably \mathcal{H}^{n-k} -rectifiable. This generalizes to this class of functions a result in [9, Sect. 4.1] valid for semiconcave (-convex) functions (see also [1]). The set Σ_∞ is not necessarily lower dimensionally rectifiable, as an example shows.

Our results are essentially based on the (local) uniqueness of the metric projection onto φ -convex sets, and use some methods taken from geometric measure theory. In some cases apparently new proofs of classical facts are given. For example, our argument, based on the area formula, for the \mathcal{H}^{n-1} -uniqueness of the unit normal vector can be applied to convex sets. Finally, we mention that regularity results, in particular double differentiability, for not necessarily Lipschitz functions were obtained in [6] for viscosity solutions of uniformly elliptic second order PDE's. Our results appear to be of a different nature, as they are derived from regularity assumptions on the epigraph rather than from an equation.

Notions of nonsmooth analysis and of geometric measure theory are recalled in Sect. 2, while the objects of our work are introduced in Sect. 3, together with some preliminary results.

2 Preliminaries

Throughout the paper, concepts of nonsmooth analysis and of geometric measure theory will be used. Although most definitions can be considered as classical, we list them in detail, in order to fix the notations. The first subsection is devoted to nonsmooth analysis, while the second one to geometric measure theory.

2.1 Nonsmooth analysis

Our environment is \mathbb{R}^n . Let $K \subseteq \mathbb{R}^n$ be closed. We denote, for $x \in \mathbb{R}^n$,

$$\begin{aligned} d_K(x) &= \min\{\|y - x\| : y \in K\} && \text{(the distance of } x \text{ from } K) \\ \pi_K(x) &= \{y \in K : \|y - x\| = d_K(x)\} && \text{(the projections of } x \text{ onto } K) \\ K_\rho &= \{y \in \mathbb{R}^n : d_K(y) \leq \rho\} \end{aligned}$$

The following simple result (see [12, p. 24]) will be often referred to:

Proposition 2.1 *Let $K \subset \mathbb{R}^n$ be nonempty, and let $x \in K$, $y \in \mathbb{R}^n$ be given. The following are equivalent:*

- (1) $x \in \pi_K(y)$;
- (2) $x \in \pi_K(x + t(y - x))$ for all $t \in [0, 1]$;
- (3) $d_K(x + t(y - x)) = t \|y - x\|$ for all $t \in [0, 1]$;
- (4) $\langle y - x, x' - x \rangle \leq \frac{1}{2} \|x' - x\|^2$ for all $x' \in K$.

Actually, for all $t \in [0, 1)$, we have $\pi_K(x + t(y - x)) = \{x\}$.

The following concepts of normals and tangents will be used (see [12, Ch. 1] and [28, Ch. 6]). Let $x \in K$ and $v \in \mathbb{R}^n$. We say that:

- 1. v is a *proximal normal* to K at x (and will be denoted by $v \in N_K^P(x)$) if there exists $\sigma = \sigma(v, x) \geq 0$ such that:

$$\langle v, y - x \rangle \leq \sigma \|y - x\|^2 \quad \text{for all } y \in K; \quad (2.1)$$

equivalently (see Proposition 2.1), $v \in N_K^P(x)$ iff there exists $\lambda > 0$ such that $\pi_K(x + \lambda v) = \{x\}$;

- 2. v is a *Fréchet normal* (or Bouligand normal) to K at x ($v \in N_K^F(x)$) if

$$\limsup_{K \ni y \rightarrow x} \left\langle v, \frac{y - x}{\|y - x\|} \right\rangle \leq 0;$$

- 3. v is a *limiting normal* to K at x ($v \in N_K^L(x)$) if

$$v \in \left\{ w : w = \lim w_n, w_n \in N_K^P(x_n), x_n \rightarrow x \right\}$$

and is a *Clarke normal* ($v \in N_K^C(x)$) if $v \in \overline{\text{co}} N_K^L(x)$;

- 4. v is a *Fréchet tangent* (or Bouligand tangent) to K at x ($v \in T_K^F(x)$) if

$$\liminf_{h \rightarrow 0^+} \frac{d_K(x + hv)}{h} = 0;$$

equivalently, $0 \neq v \in T_K^F(x)$ iff there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset K$ such that

$$\lim_{n \rightarrow \infty} \frac{y_n - x}{\|y_n - x\|} = \frac{v}{\|v\|}.$$

It can be proved (see [4, Prop. 4.4.1]) that

$$N_K^F(x) = \{v \in \mathbb{R}^n : \langle v, w \rangle \leq 0 \text{ for all } w \in T_K^F(x)\} := (T_K^F(x))^0.$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. By using $\text{epi}(f) := \{(x, \xi) : \xi \geq f(x)\}$, one can define subgradient concepts for f at $x \in \text{dom}(f) = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. Let $x \in \text{dom}(f)$, $v \in \mathbb{R}^n$. We say that:

- 1. v is a *proximal subgradient* of f at x ($v \in \partial_P f(x)$) if $(v, -1) \in N_{\text{epi}(f)}^P(x, f(x))$; equivalently (see [12, Theorem 1.2.5]), $v \in \partial_P f(x)$ iff there exist $\sigma, \eta > 0$ such that

$$f(y) \geq f(x) + \langle v, y - x \rangle - \sigma \|y - x\|^2 \quad \text{for all } y \in B(x, \eta) \cap \text{dom}(f); \quad (2.2)$$

2. v is a *Fréchet subgradient* of f at x ($v \in \partial_F f(x)$) if $(v, -1) \in N_{\text{epi}(f)}^F(x, f(x))$, i.e.,

$$\liminf_{y \rightarrow x} \frac{f(y) - f(x) - \langle v, y - x \rangle}{\|y - x\|} \geq 0;$$

3. v is a *limiting*, resp. *Clarke*, *subgradient* of f at x ($v \in \partial_L f(x)$), resp. $v \in \partial_C f(x)$) if $(v, -1) \in N_{\text{epi}(f)}^L(x, f(x))$, resp. $(v, -1) \in N_{\text{epi}(f)}^C(x, f(x))$.

Conversely, the normal concepts for sets can be deduced from the corresponding ones for functions by means of the indicator function. The inclusions

$$N_K^P(x) \subseteq N_K^F(x) \subseteq N_K^L(x) \subseteq N_K^C(x) \quad (2.3)$$

and

$$\partial_P f(x) \subseteq \partial_F f(x) \subseteq \partial_L f(x) \subseteq \partial_C f(x) \quad (2.4)$$

always hold. For a thorough analysis of the above concepts we refer to the books [12, 28].

2.2 Geometric measure theory

Let E be a subset of \mathbb{R}^n and $0 \leq k \leq n$, $k \in \mathbb{R}$. We denote by $\mathcal{L}^n(E)$ its outer Lebesgue measure, and by $\mathcal{H}^k(E)$ its k -dimensional Hausdorff measure. Among the several well known properties of Hausdorff measures, we recall (see [3, pp. 72–80]):

1. if $k > k' \geq 0$, then for every $E \subset \mathbb{R}^n$

$$\mathcal{H}^k(E) > 0 \Rightarrow \mathcal{H}^{k'}(E) = +\infty;$$

the Hausdorff dimension of a set E is $\mathcal{H} - \dim(E) = \inf \{k \geq 0 : \mathcal{H}^k(E) = 0\}$;

2. if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a Lipschitz function with Lipschitz ratio L , then for every $E \subset \mathbb{R}^n$

$$\mathcal{H}^k(f(E)) \leq L^k \mathcal{H}^k(E); \quad (2.5)$$

3. for any Borel set $B \subset \mathbb{R}^n$ one has

$$\mathcal{L}^n(B) = \mathcal{H}^n(B).$$

Let $E \subset \mathbb{R}^n$ be a \mathcal{H}^k -measurable set with $0 \leq k \leq n$, $k \in \mathbb{N}$.

We say that E is *countably k -rectifiable* if there exist countably many Lipschitz functions $f_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that

$$\mathcal{H}^k \left(E \setminus \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^k) \right) = 0.$$

We say that E is *\mathcal{H}^k -rectifiable* if E is countably \mathcal{H}^k -rectifiable and $\mathcal{H}^k(E) < +\infty$.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map. We say that f is of class $\mathcal{C}^{1,1}(\Omega)$ if it is differentiable in the open set Ω and its differential Df is Lipschitz continuous in Ω . We say that $l \in \mathbb{R}^m$ is the approximate limit of f as $y \rightarrow x$ and write $\text{ap} \lim_{y \rightarrow x} f(y) = l$ if for each $\varepsilon > 0$

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(B(x, r) \cap \{z : |f(z) - l| \geq \varepsilon\})}{\omega_n r^n} = 0$$

We recall that if the approximate limit exists then it is unique.

We say that a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is approximately continuous at $x \in \mathbb{R}^n$ if $\text{ap} \lim_{y \rightarrow x} f(y) = f(x)$.

We say that a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is approximately differentiable at $x \in \mathbb{R}^n$ if there exists a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that:

$$\text{ap} \lim_{y \rightarrow x} \frac{|f(y) - f(x) - L(y - x)|}{|y - x|} = 0$$

and write $L = \text{ap} Df(x)$.

Let $0 \leq k \leq n$, $k \in \mathbb{N}$, and let $L : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a linear map. The k -dimensional Jacobian $J_k L$ is defined to be

$$J_k L := \sqrt{\det(L^* \circ L)}$$

where $L^* : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is the transpose of L .

If $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz function, by Rademacher's theorem (see, e.g. [3, p. 47]) it is \mathcal{L}^n -a.e. differentiable. We denote by $Df(x)$ its Fréchet differential at x , which is defined \mathcal{L}^n -a.e. The following classical result will be used:

Theorem 2.1 (Area formula) *Let $k \leq n$ and let $f : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a Lipschitz function. Then, for any \mathcal{H}^k -rectifiable set $E \subset \mathbb{R}^k$ the (multiplicity) function $\mathcal{H}^0(E \cap f^{-1}(y))$, $y \in \mathbb{R}^n$, is \mathcal{H}^k -measurable on \mathbb{R}^n and*

$$\int_{f(E)} \mathcal{H}^0(E \cap f^{-1}(y)) d\mathcal{H}^k(y) = \int_E J_k D^E f(x) d\mathcal{H}^k(x),$$

where the symbol $D^E f$ denotes the tangential differential of f relative to E (see [3, Def. 2.89, p. 98]).

The following measure theoretic concept of tangent space will be considered. Let $E \subset \mathbb{R}^n$ be a Borel set with $\mathcal{H}^k(E) < +\infty$, $x_0 \in E$ and let P be a k -dimensional plane ($0 \leq k \leq n$); we say that P is the *approximate tangent space* to E in x_0 if for any $\phi \in C_c(\mathbb{R}^n)$ we have

$$\lim_{\rho \rightarrow 0^+} \rho^{-k} \int_E \phi\left(\frac{x - x_0}{\rho}\right) d\mathcal{H}^k(x) = \int_P \phi(y) d\mathcal{H}^k(y).$$

If such a k -plane P exists, then it is unique and we shall denote it with $\text{Tan}^k(E, x_0)$. The following result (see [3, pp. 96–99]) connects the existence of $\text{Tan}^k(E, x)$ with the rectifiability of E .

Proposition 2.2 *If E is k -rectifiable, then $\text{Tan}^k(E, x)$ exists for \mathcal{H}^k -a.e. $x \in E$ and:*

- (1) $\text{Tan}^k(E, x) = \text{Tan}^k(E', x)$ for \mathcal{H}^k -a.e. $x \in E \cap E'$, for each pair E, E' of rectifiable sets (localization property);
- (2) for \mathcal{H}^k -a.e. $x \in E$, $\text{Tan}^k(E, x) = T_E^F(x)$, provided E is contained in a Lipschitz graph of k variables, i.e., there exists a Lipschitz function f from a set $\Omega \subset \mathbb{R}^k$ into \mathbb{R}^{n-k} with $\mathcal{L}^k(\Omega) < +\infty$ and $E \subset \text{graph}(f)$.

The concepts of functions of bounded variation and of sets with finite perimeter will also be used (see [3, p. 117]):

1. let $\Omega \subset \mathbb{R}^n$ be open, and $u \in L^1(\Omega)$; we say that u is a *function of bounded variation in Ω* ($u \in BV(\Omega)$) if the distributional derivative of u is representable by a finite Radon measure in Ω , i.e., if

$$\int_{\Omega} u \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} \varphi dD_i u \text{ for all } \varphi \in C_c^\infty(\Omega), i = 1, \dots, n$$

for some Radon measure $Du = (D_1 u, \dots, D_n u)$. We denote by $|Du|$ the total variation of the vector measure Du .

2. Let $E \subset \mathbb{R}^n$ be \mathcal{L}^n -measurable, and let $\Omega \subseteq \mathbb{R}^n$ be open. E has *finite perimeter in Ω* if its characteristic function χ_E has bounded variation in Ω , and we say that the *perimeter* of E in Ω is

$$P(E, \Omega) = |D\chi_E|(\Omega)$$

(see [3, p. 143]).

Next we recall the following measure theoretic concept: let μ be a Radon measure on \mathbb{R}^n , and let N be the union of all open sets $U \subset \mathbb{R}^n$ such that $\mu(U) = 0$; the complement of N is called the support of μ and it is denoted by $\text{supp}(\mu)$.

The following concept of boundary will be used (see [2, Definizione 1.4.7]).

Definition 2.1 Let $E \subset \mathbb{R}^n$ be \mathcal{L}^n -measurable. We set

$$\partial_a E = \{x \in \mathbb{R}^n : \text{for all } \rho > 0, 0 < \mathcal{L}^n(E \cap B(x, \rho)) < \omega_n \rho^n\},$$

where ω_n is the n -dimensional Lebesgue measure of the unit ball in \mathbb{R}^n .

Obviously, $\partial_a E \subseteq \partial E$, where ∂E is the (topological) boundary of E , and if E is the closure of an open set then $\partial_a E = \partial E$.

The following concept of normal vector was introduced by De Giorgi. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter in Ω ; we call *reduced boundary of E in Ω* the set $\partial^* E$ of all points $x \in \text{supp}(|D\chi_E| \cap \Omega)$ such that

$$v_E(x) := \lim_{\rho \rightarrow 0^+} \frac{D\chi_E(B(x, \rho))}{|D\chi_E(B(x, \rho))|} = \frac{dD\chi_E}{d|D\chi_E|}(x)$$

exists in \mathbb{R}^n and satisfies $\|v_E(x)\| = 1$. The function $-v_E : \partial^* E \rightarrow \mathbb{R}^n$ is called the *De Giorgi outer normal* to E in x .

The following part of De Giorgi's structure theorem for sets with finite perimeter will be used (see [3, Theorem 3.59 p. 157]):

Theorem 2.2 *Let E be a set with finite perimeter in Ω . Then, for all $x \in \partial^* E$ one has:*

$$\text{Tan}^{n-1}(\partial^* E, x) = \{v_E(x)\}^\perp$$

Finally, the following measure-theoretic concepts will be used in our analysis.

Definition 2.2 Let $E \subset \mathbb{R}^n$ be a Borel set. We set, for $x \in \mathbb{R}^n$ and $0 \leq k \leq n$,

$$\delta_E^k(x) = \liminf_{\rho \rightarrow 0^+} \frac{\mathcal{H}^k(E \cap B(x, \rho))}{\omega_k \rho^k},$$

where ω_k is the k -dimensional Lebesgue measure of the unit ball in \mathbb{R}^k .

It is well known that for $k = n$ the limit actually exists for \mathcal{L}^n -a.e. $x \in E$. We denote, for $0 \leq \alpha \leq 1$, $E^\alpha = \{x \in \mathbb{R}^n : \delta_E^n(x) = \alpha\}$, and observe that, in particular, $E^{\frac{1}{2}} \subset \partial_a E$. We define now the *measure theoretic boundary* (see [3, p. 158]).

Definition 2.3 Let $E \subseteq \mathbb{R}^n$ be \mathcal{L}^n -measurable. The measure theoretic boundary of E is the set

$$\partial_M E = \mathbb{R}^n \setminus (E^0 \cup E^1).$$

Concerning the relations among the above introduced concepts of boundary, we recall the following (see [3, Theorem 3.61, p. 158]).

Theorem 2.3 (Federer) *Let E be a set of finite perimeter in Ω . Then*

$$\partial^* E \cap \Omega \subseteq E^{\frac{1}{2}} \subseteq \partial_M E \subseteq \partial_a E \subseteq \partial E$$

and

$$\mathcal{H}^{n-1}(\Omega \setminus (E^0 \cup \partial^* E \cup E^1)) = 0.$$

In particular, E has density either 0, or $\frac{1}{2}$, or 1 at \mathcal{H}^{n-1} -a.e. $x \in \Omega$, and

$$\mathcal{H}^{n-1}(\partial_M E \setminus \partial^* E) = 0.$$

The following criterion for sets of finite perimeter will be used (see [10, Theorem 4]):

Theorem 2.4 (Federer) *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $E \subseteq \Omega$ be measurable. If $\mathcal{H}^{n-1}(\partial_M(E \cap \Omega)) < +\infty$ then $P(E, \Omega) < +\infty$.*

3 φ -convex sets and functions

In this section, we introduce and illustrate in some detail the main objects of our analysis. We begin with a subsection devoted to some general definitions, and then study in more detail a subcase, which is our main interest.

3.1 General definitions

Definition 3.1 Let $K \subset \mathbb{R}^n$ be closed and let $\varphi : K \rightarrow [0, +\infty)$ be continuous. We say that K is φ -convex if for all $x, y \in K$, $v \in N_K^F(x)$, the inequality

$$\langle v, y - x \rangle \leq \varphi(x) \|v\| \|y - x\|^2 \quad (3.1)$$

holds. By φ_0 -convexity we mean φ -convexity with $\varphi \equiv \varphi_0$, a constant.

Actually (see Remark 3.2 below) it is enough to check (3.1) for $v \in N_K^P(x)$.

A detailed analysis of such sets, under the name of “sets with positive reach,” is contained in [22], where apparently this concept was stated for the first time. For related properties, in Hilbert spaces, we refer to [7, 8, 13, 15, 18, 26] and references therein.

In order to illustrate Definition 3.1, we list some simple examples, postponing to Example 4.1 a more complicated case:

1. if K is convex, then it is φ_0 -convex with $\varphi_0 = 0$;
2. if $K = \{x : g(x) \leq 0\}$, with $g \in C^1(\Omega)$ such that Dg is locally Lipschitz in Ω and $Dg(x) \neq 0$ for every $x \in \partial K$ (we will refer to such sets as to *sets with $C^{1,1}$ -boundary*), then it is φ -convex, for a suitable φ ;
3. $K = \{x = (x_1, \dots, x_n) : \max_{i=1, \dots, n} |x_i| \leq 1 \text{ and } \|x\| \geq 1\}$ is φ_0 -convex, with $\varphi_0 = \frac{1}{2}$.

Geometrically, in view of Proposition 2.1, the inequality (3.1) means that the set K satisfies a kind of external sphere condition, with locally uniform radius.

The set $K = \{(x, y) : -1 \leq y \leq |x|^{\frac{3}{2}}, |x| \leq 1\}$ is not φ -convex; actually, although ∂K is smooth around $(0, 0)$, there is no external sphere which touches K only at $(0, 0)$; accordingly the number $\sigma = \sigma(x)$ appearing in (2.1) tends to $+\infty$ as $x \rightarrow 0$.

The distance from a φ -convex set K and the metric projection onto K enjoy remarkable properties, which are fundamental for our analysis.

Theorem 3.1 Let $K \subset \mathbb{R}^n$ be a φ -convex set. Then there exists an open set $U \supset K$ such that

- (1) $d_K \in C^{1,1}(U \setminus K)$ and $Dd_K(y) = \frac{y - \pi_K(y)}{d_K(y)}$ for every $y \in U \setminus K$;
- (2) $\pi_K : U \rightarrow K$ is locally Lipschitz.

In particular, if K is φ_0 -convex (with $\varphi_0 > 0$), then $U \supset K_{\frac{1}{4\varphi_0}}$ and $\pi_K : K_{\frac{1}{4\varphi_0}} \rightarrow K$ is Lipschitz with Lipschitz ratio 2.

Proof. The proof can be found in [7, Proposition 2.6, 2.9, Remark 2.10] or in [22, Sect. 4]. \square

Remark 3.1 Conditions (1) and (2) in Theorem 3.1 are actually equivalent to φ -convexity, as it is proved, e.g., in [22, Sect. 4].

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be φ -convex. Let*

$$K_\varphi = \{x : 4d_K(x)\varphi(\pi_K(x)) < 1\}.$$

Then the set

$$\partial K_\varphi = \{x \in \mathbb{R}^n : 4d_K(x)\varphi(\pi_K(x)) = 1\}$$

is a $\mathcal{C}^{1,1}$ -manifold. In particular, it is countably \mathcal{H}^{n-1} -rectifiable. Moreover, for all $x \in \partial K_\varphi$,

$$N_{K_\varphi}^P(x) = \mathbb{R}^+(x - \pi_K(x)) \subseteq N_K^P(\pi_K(x)). \quad (3.2)$$

Proof ∂K_φ is a $\mathcal{C}^{1,1}$ -manifold because $\|Dd_K\| \equiv 1$ on ∂K_φ . Formula (3.2) is Corollary 4.15 (2) in [13]. \square

The above introduced concept has other consequences and characterizations, among which we mention (for a full list see [26]):

Proposition 3.1 *Let $K \subset \mathbb{R}^n$ be φ -convex. Then:*

- (1) *for all $x \in K$, $N_K^P(x) = N_K^F(x) = N_K^C(x)$ and the set valued map N_K^P from K into \mathbb{R}^n has closed graph; moreover, $T_K^F(x) = (N_K^C(x))^0$;*
- (2) *let $x \in K$ and let $r > 0$ be such that $4r\varphi(x) < 1$; then, for all $y_1, y_2 \in B(x, r)$ and all $t \in [0, 1]$, it holds*

$$d_K(ty_1 + (1-t)y_2) \leq 2\varphi(x)t(1-t)\|y_1 - y_2\|^2;$$

- (3) *let U be the open set enjoying the properties of Theorem 3.1; then $\pi_K(U) = \partial K$.*

Proof The proof of (1) can be found, e.g., in [15, Propositions 6.2 and 4.2] and [22, Theorem 4.8 (12)], while (2) is Proposition 2.13 in [7]. To show (3), let $x \in \partial K$ and let $0 \neq v \in N_K^C(x)$ (see [28, p. 214]); by (1), $v \in N_K^P(x)$, and this fact concludes the proof. \square

The concept of φ -convexity for sets actually can be seen as a specialization to indicator functions (see e.g. [18]) of the following definition, which appeared for the first time in [19]:

Definition 3.2 Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and let $\varphi : (\text{dom}(f))^2 \times \mathbb{R}^3 \rightarrow [0, +\infty)$ be continuous. We say that f is φ -convex if for all $x, y \in \text{dom}(f)$, for all $v \in \partial_F f(x)$ it holds that:

$$f(y) \geq f(x) + \langle v, y - x \rangle - \varphi(x, y, f(x), f(y), \|v\|)\|y - x\|^2 \quad (3.3)$$

(this inequality is automatically satisfied if $\partial_F f(x) = \emptyset$).

We say that f is φ -convex of order p ($p \geq 0$) if, for every $x, y \in \text{dom}(f)$ and $v \in \partial_F f(x)$, (3.3) holds true, and

$$\varphi(x, y, f(x), f(y), \|v\|) \leq \tilde{\varphi}(x, y, f(x), f(y))(1 + \|v\|^p)$$

for some continuous $\tilde{\varphi}$. If $\tilde{\varphi} = \varphi_0$ is constant, we say that f is φ_0 -convex of order p .

Remark 3.2 In the definition of φ -convexity, it is enough to verify the inequality (3.3) for $v \in \partial_P f(x)$.

Indeed, assume (3.3) is true for all $v \in \partial_P f(x)$, $x, y \in \text{dom}(f)$, and let $x \in \text{dom}(\partial_F f) = \{y \in \mathbb{R}^n : \partial_F f(y) \neq \emptyset\}$ and $v \in \partial_F f(x)$. Take $y \in \text{dom}(f)$. By [29, Corollary 1.11b], there exist sequences $x_n \rightarrow x$, $v_n \rightarrow v$, such that $f(x_n) \rightarrow f(x)$ and $v_n \in \partial_P f(x_n)$. By (3.3), we have:

$$f(y) \geq f(x_n) + \langle v_n, y - x_n \rangle - \varphi(x_n, y, f(x_n), f(y), \|v_n\|) \|y - x_n\|^2.$$

By passing to the limit for $n \rightarrow \infty$, we obtain that (3.3) is valid for all $v \in \partial_F f(x)$. \square

For a thorough study of this class of functions in connection with evolution equations in Hilbert spaces, see [23] and references therein. A comparison of φ -convexity with the analogous concept of *prox-regularity* (see [28, Sect. 13.F]) will be performed elsewhere. In order to illustrate the definition, we list some simple examples.

Examples of φ -convex functions:

1. a convex function is φ_0 -convex with $\varphi_0 = 0$;
2. a set K is φ -convex iff its indicator function i_K is φ -convex of order 1 (see, e.g., [15, Proposition 6.2]);
3. a $\mathcal{C}^{1,1}$ function is φ -convex;
4. $f(x) = \sqrt{|x|}$ is φ_0 -convex of order 3;
5. $f(x)$ defined by $f(x) = \sqrt{|x|}$ for $x \neq 0$ and $f(0) = -1$ is φ_0 -convex of order 3;
6. $f_\alpha(x) = |x|^\alpha \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ is φ -convex iff $\alpha \geq 3$; observe however that for $2 < \alpha < 3$, $\partial_P f_\alpha(0) = \{0\}$, so that the nonemptiness of the proximal subdifferential at every point of $\text{dom}(f)$ is not sufficient for φ -convexity;
7. $f(x) = -|x|$ and $f(x) = -|x|^{3/2}$ are not φ -convex;
8. $f(x) = \sqrt{x}$ for $x \geq 0$ and $-\sqrt{-x}$ for $x < 0$ is φ_0 -convex of order 3. Observe that $\partial_P f(0) = \emptyset$, and that both the epigraph and the hypograph of f are φ_0 -convex, but f is not $\mathcal{C}^{1,1}$;
9. $f(x) = |x|^\alpha$, $0 < \alpha < 1$, is φ_0 -convex of order $\frac{2-\alpha}{1-\alpha}$ (this can be seen by applying (3.3) with $y = 0$ and letting $x \rightarrow 0$), while $f(x) = |x|^{-\alpha}$, $x \neq 0$, $\alpha > 0$, is φ_0 -convex of order $\frac{\alpha+2}{\alpha+1}$ (this can be seen by applying (3.3) with $y = -x$ and letting $x \rightarrow 0$);
10. $f(x) = -\log |x|$, $x \neq 0$ is φ_0 -convex of order 2 (indeed, by taking $y = -x$ and letting $x \rightarrow 0^+$, (3.3) yields $2\varphi_0 x^2(1+x^{-p}) \geq 1$, which is true for $x \rightarrow 0$ iff $p \geq 2$. It is easy to see that, for $x > 0$, the parabola $z = -\log x - (y-x)/x - \varphi_0(1+x^{-p})|y-x|^2$ touches the graph of f only at $(x, f(x))$; its epigraph is φ -convex, but not φ_0 -convex;
11. $f(x) = -\cosh x$ is φ -convex of order 1, but it is not semiconvex (see Definition 3.3 below) in \mathbb{R} ; however it is semiconvex in every compact interval;
12. $f(x) = -\sqrt{|x|}$ is φ_0 -convex of order 3; observe that $\text{epi}(f)$ is not φ -convex because it is not regular, in the sense that $\mathbb{R} \times \{0\} = N_{\text{epi}(f)}^{\mathcal{C}}(0, 0) \supsetneq N_{\text{epi}(f)}^F(0, 0) = \{(0, 0)\}$;

13. $f(x) = -|x| \log |x|$ is φ -convex, but not of order p for any p (indeed, by applying (3.3) with $y = -x$ one obtains $x\varphi(|\log x + 1|) \geq -(\log x + 1)$, and this cannot be true for $x \rightarrow 0^+$ with φ of polynomial order, while it is true for, e.g., $\varphi(\xi) = e^{2\xi}$); observe that also $-f$ is φ -convex.
14. Set $R_- = \{(x_1, x_2) : x_1 > -\frac{1}{2}x_2|x_2|\}$, $R_+ = \{(x_1, x_2) : x_1 < -\frac{1}{2}x_2|x_2|\}$ and

$$f(x_1, x_2) = \begin{cases} x_2 + 2\sqrt{\frac{x_2^2}{2} + x_1}, & (x_1, x_2) \in \overline{R}_- \\ -x_2 + 2\sqrt{\frac{x_2^2}{2} - x_1}, & (x_1, x_2) \in \overline{R}_+ \end{cases}$$

Then f is φ_0 -convex of order 3 (and its epigraph is φ_0 -convex). The function f is actually the minimum time to reach the origin for the control system:

$$\begin{cases} \dot{\xi}_1 = \xi_2, & \xi_1(0) = x_1 \\ \dot{\xi}_2 \in [-1, 1], & \xi_2(0) = x_2 \end{cases}$$

(see [5, Example 2.7 p. 242]).

To show the φ_0 -convexity of $\text{epi}(f)$ it suffices to observe that the graph of f admits only downwards corners/cusps, actually along the curve $\overline{R}_- \cap \overline{R}_+$, and that cusps are of quadratic order. Then an argument similar to Example 9 above completes the proof.

An important class of nonsmooth and nonconvex functions which are φ -convex is that of semiconvex functions. The analogous class of semiconcave functions is thoroughly studied in [9].

Definition 3.3 Let Ω be an open and convex subset of \mathbb{R}^n . A function $f : \Omega \rightarrow \mathbb{R}$ is semiconvex in Ω if there exists a constant $C \geq 0$ such that for all $x_1, x_2 \in \Omega$, $\lambda \in [0, 1]$ it holds

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) + c\lambda(1 - \lambda)\|x_1 - x_2\|^2. \quad (3.4)$$

Remark 3.3 A semiconvex function is φ_0 -convex of order 0.

Indeed, according to [9, Proposition 3.3.1], if f is semiconvex, there exists $\varphi_0 \geq 0$ such that for all x, y , for all $v \in \partial_F f(x)$ one has

$$f(y) \geq f(x) + \langle v, y - x \rangle - \varphi_0 \|y - x\|^2.$$

□

Actually, the inequality (3.3) forbids “upwards corners” (i.e., corners of the type of $f(x) = |x|$ at $x = 0$) in the graph of f , analogously to semiconvex functions. However, differently from semiconvex functions, φ -convex functions need not be locally Lipschitz (see examples 4, 5, 9, 14 above). More precisely, consider a function f , φ_0 -convex of order $p > 0$. The inequality (3.3) states that it is possible to fit a parabola below the epigraph of f at any point x such that $\partial_P f(x) \neq \emptyset$; moreover the parabola touches $\text{graph}(f)$ only at $(x, f(x))$ and its axis is vertical. The key point is that its width depends on $\|v\|$, $v \in \partial_P f(x)$, and tends to zero as $\|v\|$ tends to infinity. This fact allows both downwards and upwards cusps.

A first regularity property of φ_0 -convex functions of order p is the following:

Proposition 3.2 *Let $f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ be l.s.c. and φ_0 -convex of order p . Then we have:*

$$\partial_C f(x) = \partial_L f(x) = \partial_F f(x) = \partial_P f(x)$$

for each $x \in \text{dom}(f)$.

Proof We shall prove first that if $v \in \partial_L f(x)$ then $v \in \partial_P f(x)$. By definition, there exist sequences $x_n \rightarrow x$, with $x_n \in \text{dom}(\partial_P f)$, and $v_n \in \partial_P f(x_n)$ such that $v_n \rightarrow v$. By hypothesis, for all $y \in \text{dom}(f)$, f satisfies:

$$f(y) \geq f(x_n) + \langle v_n, y - x_n \rangle - \varphi_0(1 + \|v_n\|^p) \|y - x_n\|^2$$

Passing to the liminf for $n \rightarrow +\infty$, we have that:

$$\begin{aligned} f(y) &\geq \liminf f(x_n) + \langle v, y - x \rangle - \varphi_0(1 + \|v\|^p) \|y - x\|^2 \\ &\geq f(x) + \langle v, y - x \rangle - \varphi_0(1 + \|v\|^p) \|y - x\|^2, \end{aligned}$$

namely v satisfies an inequality of the type (2.2), and hence is a proximal subgradient. Since $\partial_L f(x)$ is closed and $\partial_P f(x)$ is convex, it follows from (2.4) that $\partial_C f(x) = \partial_P f(x)$. \square

The following result is on the same line of Proposition 1.43 in [18], and shows that φ_0 -convexity of order 1 is very close to semiconvexity.

Proposition 3.3 *Let $f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ be φ_0 -convex of order 1. Let $\Omega' \subset \Omega$ be open and such that $\sup_{x \in \Omega'} f(x) := M < +\infty$. Then $\Omega' \subset \text{dom}(\partial_P(f))$ and $f|_{\Omega'}$ is locally Lipschitz.*

Proof Fix $\bar{x} \in \Omega'$ and let $\eta > 0$ be such that $\overline{B}(\bar{x}, \eta) \subset \Omega'$. Let $m = \min_{x \in \overline{B}(\bar{x}, \eta)} f(x) \in \mathbb{R}$. By the Density Theorem (see [12, Theorem 1.3.1]), $\text{dom}(\partial_P f) \cap \overline{B}(\bar{x}, \eta/2) \neq \emptyset$. We claim that there exists $K < +\infty$ such that

$$\sup \left\{ \|\xi\| : \xi \in \partial_P f(x), x \in \overline{B}\left(\bar{x}, \frac{\eta}{2}\right) \right\} \leq K. \quad (3.5)$$

Indeed, fix $x \in \text{dom}(\partial_P f) \cap \overline{B}(\bar{x}, \frac{\eta}{2})$ and $\xi \in \partial_P f(x)$. Without loss of generality, let $\xi \neq 0$. Take $0 < \bar{h} < \frac{\eta}{2}$ such that $\bar{h}\varphi_0 < 1$ and set $y_{\bar{h}} = x + \bar{h} \frac{\xi}{\|\xi\|}$. Observe that $y_{\bar{h}} \in B(\bar{x}, \eta) \subset \Omega'$. By φ_0 convexity,

$$M \geq f(y_{\bar{h}}) \geq f(x) + \bar{h}\|\xi\| - \varphi_0(1 + \|\xi\|)\bar{h}^2.$$

Therefore

$$M - m + \varphi_0\eta^2 \geq \|\xi\|(1 - \varphi_0\bar{h})\bar{h},$$

which implies (3.5).

Let $\{x_n\}_{n \in \mathbb{N}} \subset \text{dom}(\partial_P f)$ be such that $x_n \rightarrow \bar{x}$. Without loss of generality, we can also assume that there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}}$ with $\xi_n \in \partial_P f(x_n)$ and $\xi_n \rightarrow \xi$. By Proposition 3.2, $\xi \in \partial_P f(\bar{x})$ and $\|\xi\| \leq (1 - \varphi_0\bar{h})\bar{h}(M - m + \varphi_0\eta^2) := K$. By Theorem 1.7.3 in [12], f is Lipschitz of rank K on $B(\bar{x}, \frac{\eta}{2})$. \square

The following final result is geometrically evident.

Proposition 3.4 *Let $f : \overline{\Omega} \rightarrow \mathbb{R}$ be φ_0 -convex and locally Lipschitz. Then $\text{epi}(f)$ is φ_0 -convex.*

3.2 Functions with φ -convex epigraph

Our analysis will deal mostly with functions having a φ -convex epigraph. Actually, it is a natural question to compare the φ -convexity of a function with the φ -convexity of its epigraph, in analogy with convex functions. This turns out to be a rather delicate point. First of all, observe that the φ -convexity of a function does not imply, in general, the φ -convexity of its epigraph. To see this, it is enough to consider the function $f(x) = |x|^{2/3}$, which is φ_0 -convex of order 4, and observe that $v = (1, 0)$ belongs to $N_{\text{epi}(f)}^F(0, 0)$ but does not belong to $N_{\text{epi}(f)}^P(0, 0)$. A more striking example is given by $f(x) = -\sqrt{|x|}$, which is φ_0 -convex of order 3 (see the example 12 above), but its epigraph is even not regular. Actually, the φ_0 -convexity of the epigraph is stronger than the φ_0 -convexity of the function.

A simple characterization of continuous functions with φ -convex epigraph is the following:

Proposition 3.5 *Let Ω be an open subset of \mathbb{R}^n , and $f : \overline{\Omega} \rightarrow \mathbb{R}$ be continuous. Then $\text{epi}(f)$ is φ -convex if and only if the following property holds: for all $x \in \Omega$ there exist $r = r(x) > 0$ and $\varphi = \varphi(x) \geq 0$ such that for all $x_1, x_2 \in B(x, r)$, for all $\lambda \in [0, 1]$, there exists $x_\lambda \in \Omega$ such that:*

$$\|x_\lambda - (\lambda x_1 + (1 - \lambda)x_2)\| \leq \varphi \lambda (1 - \lambda) (\|x_1 - x_2\|^2 + |f(x_1) - f(x_2)|^2), \quad (3.6)$$

$$f(x_\lambda) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) + \varphi \lambda (1 - \lambda) (\|x_1 - x_2\|^2 + |f(x_1) - f(x_2)|^2). \quad (3.7)$$

Proof Recalling Propositions 2.13 in [7] and 1.12 in [8], and the fact that f is continuous, $\text{epi}(f)$ is φ -convex if and only if for all $x \in \overline{\Omega}$, there exist $\delta = \delta(x) > 0$ and $\varphi = \varphi(x) \geq 0$ such that for all $x_1, x_2 \in B(x, \delta)$, for all $\lambda \in [0, 1]$, it holds:

$$\begin{aligned} d_{\text{epi}(f)}(\lambda x_1 + (1 - \lambda)x_2, \lambda f(x_1) + (1 - \lambda)f(x_2)) &\leq \varphi \lambda (1 - \lambda) \\ &\times (\|x_1 - x_2\|^2 + |f(x_1) - f(x_2)|^2). \end{aligned} \quad (3.8)$$

Now observe that (3.6) and (3.7), together, are equivalent to (3.8), possibly with a different constant φ . \square

Remark 3.4 The above Proposition can be stated as well for f lower semicontinuous. However, the present form of (3.6) and (3.7) permits a comparison with (3.4) in the definition of semiconvexity. Actually, the main difference between semiconvexity and φ -convexity of the epigraph appears to be the lack of Lipschitz continuity for the latter case. We recall that the epigraph of a semiconvex function is always φ_0 -convex (see [9, Sect. 3.6]).

Some general regularity properties of functions with φ_0 -convex epigraph are:

Theorem 3.2 *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and such that $\text{epi}(f)$ is φ_0 -convex for some $\varphi_0 \geq 0$. Then*

- (1) f is φ_0 -convex of order 3, with a possibly different φ_0 ;
- (2) if $D = \text{dom}(f)$ is closed, then it is φ_0 -convex;
- (3) let $n = 1$; then $f \in \mathcal{L}_{\text{loc}}^\infty(D)$.

Proof (1) If $\varphi_0 = 0$, then f is convex. Thus consider $\varphi_0 > 0$ and let $\zeta \in \partial_P f(x)$. By definition, $(\zeta, -1) \in N_{\text{epi}(f)}^P(x, f(x))$. By φ_0 -convexity, for all $y \in \text{dom}(f)$

$$\langle (\zeta, -1), (y, f(y)) - (x, f(x)) \rangle \leq \varphi_0 \|(\zeta, -1)\| (\|y - x\|^2 + |f(y) - f(x)|^2).$$

Recalling Proposition 2.1, we have, for $\delta = \frac{1}{4\varphi_0\sqrt{\|\zeta\|^2+1}}$,

$$\pi_{\text{epi}(f)}((x, f(x)) + \delta(\zeta, -1)) = \{(x, f(x))\}$$

Therefore

$$\delta^2 \|(\zeta, -1)\|^2 \leq \|(y, f(y)) - (x, f(x)) - \delta(\zeta, -1)\|^2$$

for all $y \in \text{dom}(f)$, which yields

$$(f(y) - f(x) + \delta)^2 \geq \delta^2 + 2\delta \langle \zeta, y - x \rangle - \|y - x\|^2 =: g^2(y). \quad (3.9)$$

Let $\bar{\eta}$ be the unique positive solution of the equation

$$\eta^2 + 2\delta \|\zeta\| \eta - \frac{\delta^2}{2} = 0.$$

By our choice of δ , observe that $\bar{\eta} \sim \frac{\text{const}}{\varphi_0 \|\zeta\|^2}$ for $\|\zeta\| \rightarrow \infty$ and $\frac{\delta^2}{2} \leq g^2(y) \leq \delta^2 + 2\delta \|\zeta\| \bar{\eta}$ for $\|y - x\| \leq \bar{\eta}$. Recalling the definition of g in (3.9), we compute the directional derivative, for $v \in \mathbb{R}^n$,

$$g'(y; v) = \frac{\langle v, \delta\zeta - y + x \rangle}{g(y)},$$

from which it follows that, for $\|v\| \leq 1$,

$$|g(y)g'(y; v)| \leq \|\delta\zeta - y + x\|. \quad (3.10)$$

Moreover, for $v, w \in \mathbb{R}^n$,

$$g''(y; v, w) = -\frac{1}{g(y)} (\langle v, w \rangle + g'(y; w)g'(y; v)).$$

By multiplying and dividing the right hand side by $g^2(y)$ and using (3.10), we obtain for all $\|v\|, \|w\| \leq 1$, $\|y - x\| \leq \bar{\eta}$

$$|g''(y; v, w)| \leq 2\sqrt{2} \frac{g^2(y) + \delta^2 \|\zeta\|^2 + \bar{\eta}^2 + 2\delta \|\zeta\| \bar{\eta}}{\delta^3}. \quad (3.11)$$

Recalling the definition of δ and $\bar{\eta}$, the right hand side of (3.11), for $\|\zeta\| \rightarrow \infty$, is majorized by $\text{const } \varphi_0 \|\zeta\|^3$. Therefore, we have that, if $\|y - x\|$ is small enough,

$$g(y) \geq \delta + \langle \zeta, y - x \rangle - \frac{\text{const}}{2} \varphi_0 \|\zeta\|^3 \|y - x\|^2.$$

From (3.9), it follows that for $\|y - x\|$ small enough,

$$f(y) \geq f(x) + \langle \zeta, y - x \rangle - \text{const } \varphi_0 (1 + \|\zeta\|^3) \|y - x\|^2,$$

which is the desired estimate.

(2) Let $x \in D$, $v \in N_D^P(x)$, $v \neq 0$. It is easy to see that $(v, 0) \in N_{\text{epi}(f)}^P(x, \alpha)$ for all $\alpha \geq f(x)$. Take $y \in D$ and $\beta \geq f(y)$. By φ_0 -convexity of the epigraph, we have

$$\langle (v, 0), (y - x, \beta - \alpha) \rangle \leq \varphi_0 \|v\| (\|y - x\|^2 + |\beta - \alpha|^2)$$

Since there is no loss of generality in taking $\beta = \alpha$, the above inequality concludes the proof of part (2).

To show (3), take by contradiction $x \in \text{dom}(f)$ together with a sequence $x_h \rightarrow x$ such that $f(x_h) \rightarrow +\infty$. It is easy to see that the segment $S := \{(x, \alpha) : \alpha \geq f(x)\}$ lies in the boundary of $\text{epi}(f)$. Since $\text{epi}(f) \subset \mathbb{R}^2$, without loss of generality we can assume that $v = (1, 0) \in N_{\text{epi}(f)}^P(x, \alpha)$ for all $\alpha \geq f(x)$. But $d_S(x_h, f(x_h)) \rightarrow 0$. This contradicts the external sphere condition with uniform radius for v , i.e., the φ_0 -convexity of $\text{epi}(f)$. \square

Remark 3.5 With straightforward modifications, part (1) in Theorem 3.2 can be proved under the assumption that f is continuous and $\text{epi}(f)$ is φ -convex.

We present now some remarks further illustrating the definitions.

Remark 3.6 1) If f is simply φ_0 -convex in Ω , then it is not necessarily even $L^1(\Omega)$. To see this, it suffices to take $f(x) = \frac{1}{|x|}$ for $x \neq 0$, $f(0) = 0$.

2) If $\text{epi}(f)$ is φ_0 -convex, then f needs not be $W^{1,\infty}$ or $W^{2,1}$, as the example $f(x) = \sqrt{|x|}$ shows.

3) If $D = \text{dom}(f)$ is not closed, the φ -convexity of \overline{D} does not follow from the φ -convexity of $\text{epi}(f)$.

4) The necessity of the order $p = 3$ for the φ -convexity of the epigraph of a function may be understood as follows. For $x \in \text{dom}(\partial_P f)$ and $v_x \in \partial_P f(x)$, set, for $y \in \mathbb{R}^n$

$$g_x(y) = f(x) + \langle v_x, y - x \rangle - \varphi_0(1 + \|v_x\|^p) \|y - x\|^2$$

The minimum radius of curvature ρ_x at $(x, f(x))$ of the parametric surface $\{(y, z) : z = g_x(y)\}$ is

$$\frac{1}{\rho_x} = \frac{\|D^2 g_x(x)\|}{(1 + \|D g_x(x)\|^2)^{3/2}} = \frac{2\varphi_0(1 + \|v_x\|^p)}{(1 + \|v_x\|^2)^{3/2}}.$$

Observe that ρ_x is bounded away from 0 if and only if $p \leq 3$. Actually, we conjecture that if f is φ_0 -convex of order 3 and $\text{epi}(f)$ is regular, then $\text{epi}(f)$ is φ -convex. The problem to be handled in order to obtain this result is the possibility of having horizontal normal vectors to $\text{epi}(f)$. Finally, the order $p = 3$ is optimal, as it is shown by the example $f(x) = \sqrt{|x|}$.

4 Normals to φ -convex sets

In this section we show that φ -convex sets share further properties with convex sets. In particular, we show that for \mathcal{H}^{n-1} -a.e. point in ∂K , the normal cone has dimension one. Moreover, we show that if K is compact, then it has finite perimeter in \mathbb{R}^n , and if K satisfies a suitable nondegeneracy condition, then the reduced

boundary of K is \mathcal{H}^{n-1} -equivalent to $\partial_a K$ and the De Giorgi's external normal coincides \mathcal{H}^{n-1} -a.e. with the unit proximal normal. The sharpness of the nondegeneracy condition is illustrated by Example 4.1 below. The relevant definitions were recalled in Sect. 2.2 above.

The following result, due to Federer (see Remark 4.15 in [22]), will be used in the sequel.

Theorem 4.1 (Federer) *Let $K \subseteq \mathbb{R}^n$ be φ -convex, and let, for $k = 1, \dots, n$, $K^{(k)}$ be the set $\{x \in K : \mathcal{H} - \dim(N_K^P(x)) \geq n - k\}$. Then $K^{(k)}$ is a countable union of Lipschitz images of bounded k -dimensional sets, hence it is countably \mathcal{H}^k -rectifiable.*

The following result is well known if $\varphi \equiv 0$, i.e., if K is convex (see, e.g., [27, Sect. 25]).

Corollary 4.1 *Let $K \subset \mathbb{R}^n$ be φ -convex. Then, for \mathcal{H}^{n-1} -a.e. $x \in \partial K$ there exists $v_x \in \mathbb{R}^n$, $\|v_x\| = 1$, such that*

$$N_K^P(x) \subseteq \mathbb{R}v_x.$$

Proof The result is a simple consequence of Theorem 4.1. However, we like to give an alternative proof.

Let $R > 0$ be sufficiently large and set

$$\varphi_R := \max\{\varphi(x) : x \in K \cap \overline{B}(0, 2R)\}.$$

Let $0 < \rho_R < R$ be fixed such that $4\rho_R\varphi_R \leq 1$. Now for each $\lambda \in [0, 1]$ define

$$K^{R,\lambda} := \{x + \lambda\rho_R v : x \in K \cap \overline{B}(0, R), v \in N_K^P(x), \|v\| = 1\}$$

and set

$$K^R = \bigcup_{\lambda \in [0,1]} K^{R,\lambda}.$$

By Theorem 3.1, the metric projection $\pi_K : K^R \rightarrow K$ is Lipschitz continuous of rank 2. By Kirszbraun's theorem, we extend π_K to a Lipschitz function f defined on the whole of \mathbb{R}^n , with the same rank.

We set, for $R > 0$ sufficiently large,

$$E^R = \{y \in K^{R,1} : \mathcal{H}^0((f^{-1} \circ f)(y)) > 2\}.$$

We observe that $f(E^R) = \pi_K(E^R)$ consists exactly of the points $x \in \partial K \cap \overline{B}(0, R)$ where the normal cone $N_K^P(x)$ spans a subspace of dimension > 1 . In fact, since the proximal normal cone is convex, if $(x + N_K^P(x)) \cap K^{R,1}$ contains more than two points, then it contains infinitely many. In other words:

$$E^R = \{y \in K^{R,1} : \mathcal{H}^0((f^{-1} \circ f)(y)) = +\infty\}.$$

Now E^R is \mathcal{H}^{n-1} -rectifiable since $K^{R,1}$ is so (recall Corollary 3.1). By the Area Formula (see Theorem 2.1 above), one has:

$$\begin{aligned} \int_{f(E^R)} \mathcal{H}^0(f^{-1}(x) \cap E^R) d\mathcal{H}^{n-1}(x) &= \int_{E^R} J_{n-1} D^{E^R} f(y) d\mathcal{H}^{n-1}(y) \\ &\leq \text{const } \mathcal{H}^{n-1}(\partial K_{\rho_R} \cap B(0, R)) < +\infty. \end{aligned}$$

Observe that the integrand in the left hand side of the above formula is identically $+\infty$ by definition of E^R . Therefore $\mathcal{H}^{n-1}(f(E^R)) = 0$ and by the arbitrariness of R the proof is concluded. \square

Actually, it may happen that the proximal normal cone to a φ -convex set consists of exactly one line for a subset of positive \mathcal{H}^{n-1} -measure of ∂K , even if K is the closure of an open set. In other words, the boundary of a φ -convex set K may have quite a few points where the tangent cone to K has dimension less or equal to $n - 1$, or – equivalently – points with \mathcal{L}^n -density zero with respect to K . We exhibit now an example of such a behavior.

Example 4.1 A φ_0 -convex set $K \subset \mathbb{R}^2$ which is the closure of an open set, and is such that $\mathcal{H}^1(\{x \in \partial K : N_K^P(x) = \mathbb{R}v_x, \|v_x\| = 1\}) > 0$.

The following construction was inspired by the example at p. 10 of [24]. Let C be a set on the unit sphere $S = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ in \mathbb{R}^2 such that: C is closed, the interior of C in S is empty, and $\mathcal{H}^1(C) > 0$. C may be constructed as follows:

$$C = S \setminus \bigcup_{i=1}^{+\infty} I_i,$$

where I_i are open connected arcs with middle points p_i dense in S , such that $0 < \sum_{i=1}^{\infty} \mathcal{H}^1(I_i) < 2\pi$. Let B denote the open unit ball centered at the origin and let, for each $x \in C$, B_x be the open unit ball centered at x . Set

$$K = 2\overline{B} \setminus \left(B \cup \bigcup_{x \in C} B_x \right).$$

We claim that K is the closure of an open set, and K is φ_0 -convex, with $\varphi_0 = 1/2$. Indeed, let $x \in \partial K$ and assume that $\hat{x} := \pi_{\overline{B}}(x) \notin C$. Since $d_C(\hat{x}) > 0$, there exists $1 \leq r < 2$ such that $\Gamma := \{t\hat{x} : t \in (r, 2)\} \subset \text{int } K$ and x is an endpoint of the segment Γ , so that there exists a sequence of points in the interior of K which converges to x . On the other hand, if $\hat{x} \in C$, then by construction there exists a sequence of middle points p_{i_j} such that $p_{i_j} \rightarrow \hat{x}$. Therefore, for each j , there exists $1 \leq r_j < 2$ such that the segment $\{tp_{i_j} : t \in (r_j, 2)\}$ is contained in $\text{int } K$. Since necessarily $\|x\| = 2$, there exists a sequence in $\text{int } K$ converging to x .

To show that K is $1/2$ -convex, let $x \in \partial K$, and observe that there is no loss of generality in assuming that $\partial K \cap \overline{B} = \emptyset$. If $\|x\| = 2$, then, by construction, $N_K^P(x) \subseteq \mathbb{R}x$. Otherwise, x belongs to the boundary of either one or two balls (of radius 1) centered at points of C . In both cases the external sphere condition with $\varphi_0 = 1/2$ is easily seen to be satisfied for all $v \in N_K^P(x)$.

Finally, observe that $\{x \in K : N_K^P(x) = \mathbb{R}x\} \supset 2C$, and therefore it has positive \mathcal{H}^1 -measure. \square

We now posit a nondegeneracy condition on K , which will allow us to show the equivalence between $\partial_a K$ (recall Definition 2.1) and $\partial^* K$. The idea of the definition is forbidding a behavior of the type described in Example 4.1.

Definition 4.1 Let K be φ -convex. We say that K is nondegenerate if

$$\mathcal{H}^{n-1}(\{x \in \partial_a K : \delta_K^n(x) = 0\}) = 0.$$

Corollary 4.2 *Let $K \subseteq \mathbb{R}^n$ be φ -convex and nondegenerate. Then there exist $\Gamma \subset \partial_a K$ such that $\mathcal{H}^{n-1}(\partial_a K \setminus \Gamma) = 0$ and a continuous function $v : \Gamma \rightarrow \mathbb{R}^n$, $\|v(x)\| \equiv 1$, such that, for all $x \in \Gamma$,*

$$N_K^P(x) = \mathbb{R}^+ v(x).$$

Proof From Corollary 4.1 we obtain a set $\Gamma' \subset \partial K$ such that $\mathcal{H}^{n-1}(\partial K \setminus \Gamma') = 0$ and a function $v : \Gamma' \rightarrow \mathbb{R}^n$, $\|v(x)\| \equiv 1$ such that, for all $x \in \Gamma'$, $N_K^P(x) \subseteq \mathbb{R}^+ v(x)$. Since K is nondegenerate, there exists $\Gamma_0 \subset \partial_a K$ such that $\mathcal{H}^{n-1}(\Gamma_0) = 0$ and $\delta_K^n(x) > 0$ for all $x \in \partial_a K \setminus \Gamma_0$. We claim that $N_K^P(x) = \mathbb{R}^+ v(x)$ for all $x \in \Gamma' \cap (\partial_a K \setminus \Gamma_0) := \Gamma$. Indeed, if $x \in \partial K$ is such that $N_K^P(x) = \mathbb{R}v$, $\|v\| = 1$, then one has, for all $r > 0$, $K \cap B(x, r) \subseteq B(x, r) \setminus (B(x + \frac{v}{4\varphi_0}, \frac{1}{4\varphi_0}) \cup B(x - \frac{v}{4\varphi_0}, \frac{1}{4\varphi_0}))$, which implies that $\delta_K^n(x) = 0$. Recalling (1) in Proposition 3.1, $v(\cdot)$ has closed graph in $\Gamma \times \mathbb{R}^n$. Since v is uniformly bounded, it is continuous in Γ . \square

The next result concerns the density of K at boundary points with unique unit proximal normal. It will be needed in the comparison among some nonsmooth analysis and geometric measure theory objects at the end of this section.

Proposition 4.1 *Let $K \subset \mathbb{R}^n$ be φ -convex and let $x \in K$ be such that $N_K^P(x) = \mathbb{R}^+ v$, with $\|v\| = 1$. Then*

$$\delta_K^n(x) = \frac{1}{2}.$$

Proof (a) We prove first that $\delta_K^n(x) \geq \frac{1}{2}$. Since $N_K^P(x) = N_K^F(x) = N_K^C(x)$ for all x (see Proposition 3.1 (1)), we have that $T_K^F(x) = (N_K^P(x))^0 = (\mathbb{R}^+ v)^0$ (see [28, p. 220]). Therefore $T_K^F(x)$ is a n -dimensional half space, whence

$$\mathcal{L}^n(T_K^F(x) \cap B(0, r)) = \frac{1}{2} \omega_n r^n$$

for all $r > 0$. Recalling [22, 4.15 (2)], we then have

$$1 \leq \liminf_{r \rightarrow 0^+} \frac{\mathcal{L}^n(K \cap B(x, r))}{\mathcal{L}^n(T_K^F(x) \cap B(0, r))} = \liminf_{r \rightarrow 0^+} \frac{\mathcal{L}^n(K \cap B(x, r))}{\frac{1}{2} \omega_n r^n} = 2\delta_K^n(x),$$

which proves our claim.

(b) Next we show that $\limsup_{r \rightarrow 0^+} \frac{\mathcal{L}^n(K \cap B(x, r))}{\omega_n r^n} \leq \frac{1}{2}$. Indeed, $K \cap B(x, r) \subseteq B(x, r) \setminus B(x + \frac{v}{4\varphi_0}, \frac{1}{4\varphi_0})$ and $\mathcal{L}^n(B(x, r) \setminus B(x + \frac{v}{4\varphi_0}, \frac{1}{4\varphi_0})) \sim \frac{1}{2} \omega_n r^n$ for $r \rightarrow 0^+$. \square

Corollary 4.3 *Let K be φ -convex. Then for \mathcal{H}^{n-1} -a.e. $x \in \partial K$, either $\delta_K^n(x) = 0$ or $\delta_K^n(x) = \frac{1}{2}$. In particular, if $K = \overline{\Omega}$, Ω open, is nondegenerate, then for \mathcal{H}^{n-1} -a.e. $x \in \partial K$, we have that $\delta_K^n(x) = \frac{1}{2}$.*

Proof It is enough to observe that if $N_K^P(x) = \mathbb{R}^+ v$, $\|v\| = 1$, then the argument used in the proof of Corollary 4.2 yields that $\delta_K^n(x) = 0$. The result then follows from Proposition 4.1 and Corollary 4.2. \square

Theorem 4.2 *Let $K \subset \mathbb{R}^n$ be compact and φ_0 -convex. Then ∂K is \mathcal{H}^{n-1} -rectifiable and K has finite perimeter in \mathbb{R}^n .*

Proof Recalling Corollary 3.1, if $\varphi_0\rho < \frac{1}{4}$, ∂K_ρ is \mathcal{H}^{n-1} -rectifiable. Moreover $\pi_K : K_\rho \rightarrow \partial K$ is Lipschitz continuous and onto (see (3) in Proposition 3.1), which proves the \mathcal{H}^{n-1} -rectifiability of ∂K . By Theorem 2.4, since $\partial_M K \subset \partial K$, the \mathcal{H}^{n-1} -rectifiability of ∂K implies immediately that $P(K, \mathbb{R}^n) < +\infty$. \square

With the same argument, we obtain also:

Proposition 4.2 *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and such that $\text{epi}(f)$ is φ -convex. Then for all open and bounded $U \subset \mathbb{R}^{n+1}$, $P(\text{epi}(f), U) < +\infty$.*

It is now our aim to prove a statement which implies that if K is the closure of an open set and is nondegenerate, then $\mathcal{H}^{n-1}(\partial K \setminus \partial^* K) = 0$. We recall that, in general, this property does not hold (see, e.g., [3, Example 3.5.3 p.154]).

Theorem 4.3 *Let $K \subset \mathbb{R}^n$ be compact and φ_0 -convex. Then for every $x \in \partial^* K$ there exists v_x , $\|v_x\| = 1$, such that*

$$N_K^P(x) = \mathbb{R}^+ v_x. \quad (4.1)$$

Furthermore, for \mathcal{H}^{n-1} -a.e. $x \in \partial^* K$,

$$v_x = -v_K(x). \quad (4.2)$$

Finally, if K is nondegenerate, then

$$\mathcal{H}^{n-1}(\partial_a K \setminus \partial^* K) = 0.$$

Proof Let $x \in \partial^* K$ and assume, by contradiction, that there exist $v_1, v_2 \in N_K^P(x)$ with $\|v_1\| = \|v_2\| = 1$ and $\mathbb{R}^+ v_1 \neq \mathbb{R}^+ v_2$. By the external sphere condition (arguing as in the proof of part (b) of Proposition 4.1) it is easy to see that $\delta_K^n(x) < \frac{1}{2}$. Since $\partial^* K \subset K^{1/2}$ (recall Theorem 2.3) this is impossible, hence (4.1) is proved.

To show (4.2), observe first that, since ∂K is \mathcal{H}^{n-1} -rectifiable, we can find at most countably many $(n-1)$ -Lipschitz graphs $\{\Gamma_i\}$ such that $\mathcal{H}^{n-1}(\partial K \setminus \bigcup_i \Gamma_i) = 0$. Fix $x \in \partial^* K \cap \Gamma_{\bar{i}}$, for some \bar{i} . By possibly dropping a further set of \mathcal{H}^{n-1} -measure 0, by Proposition 2.2 we obtain that

$$\text{Tan}^{n-1}(\partial^* K, x) = \text{Tan}^{n-1}(\partial K \cap \Gamma_{\bar{i}}, x) = T_{\partial K \cap \Gamma_{\bar{i}}}^F(x) \subseteq T_{\partial K}^F(x) \subseteq T_K^F(x). \quad (4.3)$$

By Theorem 2.2 and (4.3), we obtain

$$\{v_K(x)\}^\perp \subseteq T_K^F(x).$$

By taking polars and recalling formula (4.1) and (1) in Proposition 3.1, we have that $\mathbb{R}^+ v_x = N_K^P(x) \subseteq \mathbb{R} v_K(x)$, from which it follows that $v_x = \pm v_K(x)$. Recalling [20, Corollary 1, p. 203], we have that

$$\lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(B(x, r) \cap K \cap H^+(x))}{r^n} = 0, \quad (4.4)$$

where $H^+(x) = \{y \in \mathbb{R}^n : \langle y - x, -v_K(x) \rangle \geq 0\}$. By the φ_0 -convexity of K , property (4.4) holds as well if, in the definition of $H^+(x)$, $-v_K(x)$ is substituted by v_x . If $v_x = v_K(x)$, then (4.4) would hold with $H^-(x) = \{y \in \mathbb{R}^n : \langle y - x, v_K(x) \rangle \geq 0\}$ in place of $H^+(x)$. As a consequence, $\delta_K^n(x) = 0$, which would contradict Proposition 4.1. The proof of (4.2) is therefore concluded.

Let now K be nondegenerate. By Corollary 4.3, $\mathcal{H}^{n-1}(\partial_a K \setminus K^{1/2}) = 0$. By Theorem 2.3, $\mathcal{H}^{n-1}(\partial_M K \setminus \partial^* K) = 0$. Therefore $\mathcal{H}^{n-1}(\partial_a K \setminus \partial^* K) = 0$. \square

Having equality between Clarke and proximal normals, we derive a regularity property of the approximate tangent space \mathcal{H}^{n-1} -a.e. on the topological boundary of K .

Corollary 4.4 *Let $K \subset \mathbb{R}^n$ be compact and φ_0 -convex. Then for \mathcal{H}^{n-1} -a.e. $x \in \partial K$, the following property holds:*

$$\lim_{K \ni x' \rightarrow x, h \rightarrow 0} \frac{d_K(x' + hv)}{h} = 0$$

uniformly for $v \in \text{Tan}^{n-1}(\partial K, x)$, $\|v\| = 1$.

Proof Recalling Theorem 3.1 (2), Proposition 2.2 (2), and Proposition 3.1 (1) we have that, for \mathcal{H}^{n-1} -a.e. $x \in \partial K$, $\text{Tan}^{n-1}(\partial K, x) = T_{\partial K}^F(x) \subseteq T_K^F(x) = (N_K^P(x))^0 = (N_K^C(x))^0$. Now the half space $(N_K^C(x))^0$ is the (Clarke) tangent cone to K , i.e. the set of vectors v such that

$$\lim_{K \ni x' \rightarrow x, h \rightarrow 0} \frac{d_K(x' + hv)}{h} = 0 \quad (4.5)$$

(see [12, Proposition 2.5.2]). Since $\{v \in \text{Tan}^{n-1}(\partial K, x) : \|v\| = 1\}$ is compact, it is easy to see that the limit in (4.5) is uniform with respect to $v \in \text{Tan}^{n-1}(\partial K, x)$, $\|v\| = 1$. \square

5 Differentiability of functions with φ -convex epigraph

This section is devoted to the \mathcal{L}^n -a.e. differentiability of l.s.c. functions with φ -convex epigraph. Our main result is the following.

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be open, and let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and such that $\text{epi}(f)$ is φ -convex. Then there exists a sequence of sets $\Omega_h \subseteq \Omega$ such that Ω_h is compact in $\text{dom}(f)$ and*

(1) *the union of Ω_h covers \mathcal{L}^n -almost all $\text{dom}(f)$, i.e.,*

$$\mathcal{L}^n\left(\text{dom}(f) \setminus \bigcup_h \Omega_h\right) = 0; \quad (5.1)$$

(2) *for all $x \in \bigcup_h \Omega_h$ there exist $\delta = \delta(x) > 0$, $L = L(x) > 0$ such that*

$$f \text{ is Lipschitz on } B(x, \delta) \text{ with ratio } L; \quad (5.2)$$

(3) for all $x \in \bigcup_h \Omega_h$

f is (strictly) Fréchet differentiable at x ; (5.3)

(4) for all $x \in \bigcup_h \Omega_h$, for all sequences $\{y_m\} \subset \text{dom}(Df)$ such that $y_m \rightarrow x$, we have

$$Df(y_m) \rightarrow Df(x); \quad (5.4)$$

(5) for all $x \in \bigcup_h \Omega_h$,

Df is approximately continuous at x . (5.5)

The idea of the proof is the following. By Corollary 4.1, for \mathcal{H}^n -a.e. point (x, α) in the boundary of the epigraph of f the normal cone to $\text{epi}(f)$ has dimension one. The point of the proof is showing that for \mathcal{L}^n -a.e. x this normal cone is not horizontal, and hence the Clarke subdifferential is nonempty and is a singleton. This is achieved by exploiting the property that the boundary of a suitable neighborhood of the epigraph is a $\mathcal{C}^{1,1}$ -manifold, and therefore the projection onto \mathbb{R}^n of the points in it with horizontal normal has \mathcal{L}^n -measure zero.

Proof of Theorem 5.1. We write, for $y \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $\pi_{\mathbb{R}^n}(y, \xi) = y$ and denote by $\{e_i\}$ the canonical basis in \mathbb{R}^{n+1} . We denote also $x \in \mathbb{R}^{n+1}$ as $x = (y, \xi) = (y_1, \dots, y_n, \xi)$.

Set $K = \text{epi}(f)$ and let $R > 0$ be sufficiently large. Set $\varphi_R = \max\{\varphi(x) : x \in K \cap \overline{B}(0, R)\}$ and take $0 < \rho_R < R$ such that $4\varphi_R \rho_R < 1$. Set, for $\lambda \in [0, 1]$,

$$K^{R,\lambda} = \{x + \lambda \rho_R v : x \in K \cap \overline{B}(0, R), v \in N_K^P(x), \|v\| = 1\}.$$

Set also

$$K^R = \bigcup_{\lambda \in [0,1]} K^{R,\lambda},$$

and

$$\tilde{K}^R = K^{R,1}.$$

Observe that, by Theorem 3.1 and Corollary 3.1, d_K is of class $\mathcal{C}^{1,1}$ in a neighborhood of \tilde{K}^R , and \tilde{K}^R is a $\mathcal{C}^{1,1}$ -manifold. Moreover, again by Theorem 3.1,

$$\pi_K \text{ is Lipschitz with rank 2 on } \tilde{K}^R. \quad (5.6)$$

We now observe the following:

Claim 1: Set for $1 > \varepsilon \geq 0$

$$E_\varepsilon^R = \left\{ (y, \xi) \in \tilde{K}^R : \left| \frac{\partial}{\partial \xi} d_K(y, \xi) \right| \leq \varepsilon \right\}.$$

Then there exists a constant C_R independent of ε such that

$$\mathcal{L}^n(\pi_{\mathbb{R}^n}(E_\varepsilon^R)) \leq C_R \varepsilon$$

for all $\varepsilon \geq 0$.

Proof of Claim 1. Take $(\bar{y}, \bar{\xi}) \in E_\varepsilon^R$. Since $\|Dd_K\| \equiv 1$, we can assume that $\frac{\partial d_K}{\partial y_i}(\bar{y}, \bar{\xi}) \neq 0$ for some i . We treat the case $i = 1$, the others being handled symmetrically.

By the implicit function theorem, there exist an open cube $U_1 \subset \mathbb{R}^n$ centered at $(\bar{y}_2, \dots, \bar{y}_n, \bar{\xi})$ and an interval V_1 centered at \bar{y}_1 , a \mathcal{C}^1 function $\psi : U_1 \rightarrow V_1$ and a constant $c_1 \geq 0$ such that

$$\psi(\bar{y}_2, \dots, \bar{y}_n, \bar{\xi}) = \bar{y}_1, \quad \tilde{K}^R \cap (V_1 \times U_1) = \tilde{\psi}(U_1),$$

where

$$\tilde{\psi}(y_2, \dots, y_n, \xi) = (\psi(y_2, \dots, y_n, \xi), y_2, \dots, y_n, \xi),$$

and

$$\left| \frac{\partial \psi}{\partial \xi}(y_2, \dots, y_n, \xi) \right| = \left| \frac{\frac{\partial d_K}{\partial \xi}(\tilde{\psi}(y_2, \dots, y_n, \xi))}{\frac{\partial d_K}{\partial y_1}(\tilde{\psi}(y_2, \dots, y_n, \xi))} \right| \leq c_1 \varepsilon \quad (5.7)$$

for all $(y_2, \dots, y_n, \xi) \in U_1$ such that $(\psi(y_2, \dots, y_n, \xi), y_2, \dots, y_n, \xi) \in E_\varepsilon^R$. Observe that the constant c_1 and the sets U_1 and V_1 can be chosen independently of ε , for ε small.

Now observe that, by (5.7) and (2.5),

$$\begin{aligned} \mathcal{L}^n(\pi_{\mathbb{R}^n}(\tilde{\psi}(U_1) \cap E_\varepsilon^R)) &= \int_{\tilde{\psi}^{-1}(E_\varepsilon^R)} \left| \frac{\partial \psi}{\partial \xi}(y_2, \dots, y_n, \xi) \right| dy_2 \dots dy_n d\xi \\ &\leq c_1 \varepsilon \mathcal{L}^n(U_1). \end{aligned}$$

Hence there exists c_2 , independent of ε , such that

$$\mathcal{L}^n(\pi_{\mathbb{R}^n}(E_\varepsilon^R \cap (V_1 \times U_1))) \leq c_2 \varepsilon. \quad (5.8)$$

By compactness, E_ε^R can be covered by finitely many cubes $U_i \times V_i$, each of them satisfying an inequality of the type (5.8). Moreover, the maximal number of those cubes does not depend on ε , for ε small. This completes the proof of the claim.

As a consequence of Claim 1, observe that

$$\mathcal{L}^n \left(\bigcup_{R>0} \left\{ y : (y, \xi) \in \tilde{K}^R, \frac{\partial}{\partial \xi} d_K(y, \xi) = 0 \right\} \right) = \mathcal{L}^n \left(\bigcup_{R>0} \pi_{\mathbb{R}^n}(E_0^R) \right) = 0. \quad (5.9)$$

Consider now the set

$$K_0 = \{(x, \xi) \in K : N_K^P(x, \xi) \subseteq \mathbb{R}v, \|v\| = 1, \text{ and } \langle v, e_{n+1} \rangle = 0\}.$$

We have

Claim 2: $\mathcal{L}^n(\pi_{\mathbb{R}^n}(K_0)) = 0$.

Proof of Claim 2. By definition of K_0 , for all $(x, \xi) \in K_0$ there exist $R = R(x) > 0$, $\lambda = \lambda(x) > 0$ and $w = w(x) \in \mathbb{R}^n$, $\|w\| = 1$, such that

1. $(x, f(x)) \in B(0, R)$,

2. $(w, 0) \in N_K^P(x, f(x))$,
3. $(x + \lambda w, f(x)) := (y, f(x)) \in E_0^R$.

Fix $\varepsilon > 0$. By (5.9) we can cover $\pi_{\mathbb{R}}^n(E_0^R)$ by countably many balls $B_i = B(z_i, r_i)$ such that

$$\sum_i \mathcal{H}^{n-1}(\partial B_i) < \varepsilon.$$

Define the cylinders

$$C_i = \{(z, \xi) : z \in \partial B_i, |\xi| \leq R\}.$$

Observe that there is no loss of generality in taking the r_i so small that $4\varphi_R(\rho_R + 2r_i) < 1$. Therefore, recalling (3.2), there exists $i \in \mathbb{N}$, $\sigma_x \geq 0$ such that $y \in B_i$, $(y + \sigma_x w, f(x)) \in C_i \cap \tilde{K}_R$, and

$$\pi_K(y + \sigma_x w, f(x)) = \{(x, f(x))\}.$$

So, for all $(x, \xi) \in K_0$ there exists $R > 0$ such that

$$\pi_{\mathbb{R}^n}(x, \xi) = \pi_{\mathbb{R}^n}(x, f(x)) \subset (\pi_{\mathbb{R}^n} \circ \pi_K) \left(\bigcup_i C_i \cap \tilde{K}_R \right).$$

Recalling (5.6), $\pi_{\mathbb{R}^n} \circ \pi_K$ is Lipschitz with ratio 2 on each $C_i \cap \tilde{K}_R$. Therefore

$$\mathcal{L}^n \left((\pi_{\mathbb{R}^n} \circ \pi_K) \left(\bigcup_i C_i \cap \tilde{K}_R \right) \right) \leq 2^n \mathcal{H}^n \left(\bigcup_i C_i \right) \leq 2^{n+1} \varepsilon R.$$

Hence

$$\mathcal{L}^n(\pi_{\mathbb{R}^n}(K_0 \cap B(0, R))) = 0.$$

Claim 2 then follows immediately.

We now observe the following:

Claim 3: $\mathcal{L}^n(\text{dom}(f) \cap \partial \text{dom}(f)) = 0$.

Proof of Claim 3. Let $y \in \text{dom}(f) \cap \partial \text{dom}(f)$. Only two cases may occur:

1. $N_K^P(y, f(y)) \subseteq \mathbb{R}\zeta_y$ with $\zeta_y = (v_y, 0)$, $\|v_y\| = 1$;
2. $N_K^P(y, f(y))$ has dimension > 1 .

In fact, if $N_K^P(y, f(y)) = \mathbb{R}^+ \zeta_y$, then $\langle \zeta_y, e_{n+1} \rangle = 0$. Indeed, since $y \in \text{dom}(f) \cap \partial \text{dom}(f)$, for all $\alpha > f(y)$, $(y, \alpha) \in \partial K$. Therefore, for all $\zeta_\alpha \in N_K^P(y, \alpha)$, $\|\zeta_\alpha\| = 1$, we have $\langle \zeta_\alpha, e_{n+1} \rangle = 0$. By the regularity of the epigraph, it follows that $\zeta_y = \lim_{\alpha \rightarrow f(y)^+} \zeta_\alpha$ is horizontal. Now observe that if $N_K^P(y, f(y)) = \mathbb{R}\zeta_y$, then obviously $\langle \zeta_y, e_{n+1} \rangle = 0$. Recalling Claim 2 and Corollary 4.1 the claim now follows.

To conclude the proof of the theorem, assume that $\mathcal{L}^n(\text{dom}(f) \setminus \partial \text{dom}(f)) > 0$ (otherwise there is nothing to prove). For all $\varepsilon > 0$ let

$$K_\varepsilon = \{(y, f(y)) : N_K^P(y, f(y)) \subseteq \mathbb{R}v_y, \|v_y\| = 1 \text{ and } |\langle v_y, e_{n+1} \rangle| < \varepsilon\}.$$

Recalling Claim 2,

$$\mathcal{L}^n \left(\bigcap_{\varepsilon > 0} \pi_{\mathbb{R}^n}(K_\varepsilon) \right) = 0. \quad (5.10)$$

By Claim 3 and Corollary 4.1, let F_ε be an open subset of \mathbb{R}^n such that

$$\{y \in \text{dom}(f) : \dim N_K^P(y, f(y)) > 1\} \cup (\text{dom}(f) \cap \partial \text{dom}(f)) \cup (\pi_{\mathbb{R}^n}(K_\varepsilon)) \subset F_\varepsilon$$

and

$$\mathcal{L}^n(F_\varepsilon) < \mathcal{L}^n(\pi_{\mathbb{R}^n}(K_\varepsilon)) + \varepsilon.$$

Set, for $h \in \mathbb{N}$,

$$\Omega_h = \{y \in \text{dom}(f) : d_{\partial \Omega}(y) \geq 1/h, \|y\| \leq h\} \setminus F_{1/h}.$$

For h large enough, Ω_h is a nonempty subset of Ω , which is contained in the interior of $\text{dom}(f)$ and is compact in $\text{dom}(f)$. By construction we have

$$\mathcal{L}^n \left(\text{dom}(f) \setminus \bigcup_{h \in \mathbb{N}} \Omega_h \right) = 0.$$

We claim that f is strictly Fréchet differentiable at all $y \in \bigcup_h \Omega_h$. Indeed, for all $y \in \bigcup_h \Omega_h$, say $y \in \Omega_h$, the cone $N_K^P(y, f(y))$ is a half line, $N_K^P(y, f(y)) = \mathbb{R}^+ v_y$, $\|v_y\| = 1$, and $|\langle v_y, e_{n+1} \rangle| \geq \frac{1}{h}$. Therefore the proximal subgradient $\partial_P f(y)$ is nonempty and is actually a singleton, say $\partial_P f(y) = \zeta_y$. The differentiability of f at x , then, will follow from the next claim:

Claim 4. Let $x \in \text{int dom}(f)$ be such that $\partial_P f(x)$ is a singleton. Then f is (strictly) Fréchet differentiable at x .

Proof of Claim 4. By assumption there exists $v_x \in \mathbb{R}^n$, $\|v_x\| = 1$, such that both $N_K^P(x, f(x)) = \mathbb{R}^+ v_x$ and

$$\langle v_x, e_{n+1} \rangle \neq 0. \quad (5.11)$$

We observe first that f is continuous at x . For, should a sequence $y_n \rightarrow x$, $y_n \in \text{int dom}(f)$, exist such that $\lim f(y_n) := \bar{\xi} > f(x)$, the segment joining $(x, f(x))$ with $(x, \bar{\xi})$ would be in the boundary of $K = \text{epi}(f)$. Then, all $v \in N_K^P(x, \alpha)$ for all $f(x) < \alpha \leq \bar{\xi}$ ought to satisfy $\langle v, e_{n+1} \rangle = 0$ by the external sphere condition. By the regularity of the epigraph (recall Proposition 3.1 (1)) and the fact that $N_K^P(x, f(x)) = \mathbb{R}^+ v_x$, we would obtain $\langle v_x, e_{n+1} \rangle = 0$, a contradiction with (5.11). Therefore, by the regularity of the epigraph, there exist $\delta = \delta(x) > 0$ and $\eta = \eta(x) > 0$ such that $B(x, \delta) \subset \Omega$ and

$$\|y - x\| < \delta \text{ implies } N_K^P(y, f(y)) \cap \{w : \|w\| = 1\} \subseteq \{w : |\langle w, e_{n+1} \rangle| \geq \eta\}.$$

This shows that the proximal subgradient of f is nonempty and uniformly bounded in $B(x, \delta)$. By Theorem 7.3, p. 52 in [12], f is Lipschitz in $B(x, \delta)$. By the regularity of the epigraph, $\partial_C f(x) = \partial_P f(x) = \{\zeta_x\}$. Thus, by Proposition 2.2.4 in [11], f is strictly Fréchet differentiable at x , and Claim 4 is proved.

Now, (5.4) follows immediately from the regularity of the epigraph of f , and (5.5) follows from (5.4) and the local boundedness of Df around any $y \in \bigcup_h \Omega_h$. The proof of Theorem 5.1 is concluded. \square

- Remark 5.7* 1) Recalling Proposition 4.2 and Theorem 4.3, our estimate in Claim 2 is in the spirit of a theorem by Vol’pert, as presented in [10, Theorem G]. This result affirms essentially that, given a set E of finite perimeter in \mathbb{R}^n , the projection onto \mathbb{R}^{n-1} of the set $\{x \in \partial^* E : \langle \nu_E(x), e_n \rangle = 0\}$ has \mathcal{L}^{n-1} -measure zero. However, observe that $\text{epi}(f)$ has not necessarily finite perimeter in \mathbb{R}^n . Moreover, we deal with the topological boundary of $\text{epi}(f)$ rather than with the reduced boundary.
- 2) Claim 2 in the above proof can be seen as a regularity property of $\text{dom}(f)$. Observe that both in Claim 1 and in Claim 2 the fact that f is defined on the closure of an open set was not used.

We conclude the section by studying the nondifferentiability set of f . We set

$$\begin{aligned}\Sigma(f) &= \{x \in \text{int dom}(f) : f \text{ is not differentiable at } x\}, \\ \Sigma_k(f) &= \{x \in \text{int dom}(f) : \mathcal{H} - \dim(\partial_P f(x)) \geq k\}, \\ \Sigma_\infty(f) &= \{x \in \text{int dom}(f) : \partial_P f(x) = \emptyset\}.\end{aligned}$$

By the analysis in the proof of Theorem 5.1 (see Claim 4 above), we have

$$\Sigma(f) = \bigcup_{k=1}^n \Sigma_k(f) \cup \Sigma_\infty(f).$$

By Theorem 5.1, $\mathcal{L}^n(\Sigma(f)) = 0$. It is a natural question studying the Hausdorff dimension of $\Sigma(f)$. The sets $\Sigma_k(f)$ and $\Sigma_\infty(f)$ are considered separately. The following partial generalization of Corollary 4.1.13 in [9] holds.

Proposition 5.1 *Let the assumptions of Theorem 5.1 be satisfied. Then, for every $k = 1, \dots, n$ the set $\Sigma_k(f)$ is countably \mathcal{H}^{n-k} -rectifiable.*

Proof Observe that $\Sigma_k = \pi_{\mathbb{R}^n}(\{(x, \xi) \in \text{epi}(f) : \dim(N_{\text{epi}(f)}^P(x, \xi)) \geq k+1\})$. Recalling Theorem 4.1, the set $\{(x, \xi) \in \text{epi}(f) : \dim(N_{\text{epi}(f)}^P(x, \xi)) \geq k+1\}$ is countably $\mathcal{H}^{n+1-(k+1)}$ -rectifiable. By the Lipschitzianity of the canonical projection, $\Sigma_k(f)$ is \mathcal{H}^{n-k} -rectifiable. \square

In general, the Hausdorff dimension of Σ_∞ may be arbitrarily close to n , as the following example shows.

Example 5.2 For each $0 < d < 1$ there exist $\alpha_d > 0$ and a function $f : [0, \alpha_d] \rightarrow [0, 1]$ with φ -convex epigraph and such that $\mathcal{H} - \dim(\Sigma_\infty(f)) = d$. In particular, $\Sigma_\infty(f)$ is uncountable.

Let C be a Cantor set in $[0, 1]$ with $\mathcal{H} - \dim(C) = d$ (see [21, Example 4.5, p. 58]). In particular, C is closed and totally disconnected. Set

$$g(x) = \int_0^x d_C(t) dt, \quad x \in [0, 1].$$

Then g is in $\mathcal{C}^{1,1}([0, 1])$ and is strictly increasing (for, if $x_1 < x_2$ there exists a set of positive measure contained in $(x_1, x_2) \setminus C$). In particular, the set $g(C) = g(\{x : g'(x) = 0\})$ is uncountable.

Now let $f(x) := g^{-1}(x)$, for $x \in [0, g(1)]$. Its graph is the symmetric of $\text{graph}(g)$ w.r.t. $\{(x, x) : x \geq 0\}$; in particular, it is a $\mathcal{C}^{1,1}$ -curve. Therefore, $\text{epi}(f)$ is φ_0 -convex. Observe that $\Sigma_\infty(f) = C$. \square

6 Second order properties

In this section, we show that the class of functions considered in Sect. 5 is not very far from semiconvex functions.

Theorem 6.1 *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and such that $\text{epi}(f)$ is φ -convex. Then for \mathcal{L}^n -a.e. $x \in \text{dom}(f)$, there exists $\delta = \delta(x) > 0$ such that f is semiconvex on $B(x, \delta)$. More precisely, let $\{\Omega_h\}$ be a sequence of sets contained in Ω satisfying (5.1) and (5.2), and define*

$$A := \bigcup_h \Omega_h. \quad (6.1)$$

Then, for all $x \in A$ there exists $\delta(x) > 0$ such that $f|_{B(x, \delta(x))}$ is semiconvex.

Proof Let $x \in A$ and fix $0 < \eta = \eta(x) < 1$ such that $B(x, \eta) \subset \Omega$ and there exists $L = L(x) > 0$ such that f is Lipschitz on $\overline{B}(x, \eta)$ with ratio L . Set $\varphi_0 = \max\{\varphi(y, f(y)) : y \in \overline{B}(x, \eta)\}$. Let $0 < \delta < \frac{\eta}{2}$ be such that $\varphi_0(1 + L^2)\delta < \eta$. Take $x_1, x_2 \in B(x, \delta)$. By the φ -convexity of $\text{epi}(f)$, recalling (2) in Proposition 3.1, we have:

$$\begin{aligned} d_{\text{epi}(f)}\left(\frac{x_1 + x_2}{2}, \frac{f(x_1) + f(x_2)}{2}\right) &\leq \frac{\varphi_0}{2}(\|x_1 - x_2\|^2 + |f(x_1) - f(x_2)|^2) \\ &\leq \frac{\varphi_0}{2}(\|x_1 - x_2\|^2 + L^2\|x_1 - x_2\|^2) \quad (6.2) \end{aligned}$$

$$\leq \frac{\varphi_0}{2}(1 + L^2)\delta^2 \leq \frac{\eta}{2}, \quad (6.3)$$

where the last inequality follows from our choice of $\delta < 1$. By (6.2) and (6.3) there exists $\bar{x} \in B(x, \eta)$ such that

$$\left\| \bar{x} - \frac{x_1 + x_2}{2} \right\| + \left| f(\bar{x}) - \frac{f(x_1) + f(x_2)}{2} \right| \leq \frac{\varphi_0}{\sqrt{2}}(1 + L^2)\|x_1 - x_2\|^2.$$

In particular

$$\left\| \bar{x} - \frac{x_1 + x_2}{2} \right\| \leq \frac{\varphi_0}{\sqrt{2}}(1 + L^2)\|x_1 - x_2\|^2,$$

and

$$\left| f(\bar{x}) - \frac{f(x_1) + f(x_2)}{2} \right| \leq \frac{\varphi_0}{\sqrt{2}}(1 + L^2)\|x_1 - x_2\|^2.$$

Therefore,

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2}\right) &\leq f(\bar{x}) + L \left\| \frac{x_1 + x_2}{2} - \bar{x} \right\| \\ &\leq \frac{f(x_1) + f(x_2)}{2} + \frac{\varphi_0}{\sqrt{2}}(1 + L^2)(1 + L)\|x_1 - x_2\|^2, \end{aligned}$$

which implies the semiconvexity of f on $B(x, \delta)$. \square

Corollary 6.1 *Under the same assumptions of Theorem 6.1, f is \mathcal{L}^n -a.e. twice differentiable on Ω , in the sense that for \mathcal{L}^n -a.e. $x \in \Omega$ there exists a symmetric $n \times n$ matrix X_x such that*

$$Df(y) = Df(x) + X_x(y - x) + o(\|y - x\|)$$

for $y \rightarrow x$, $y \in \text{dom}(Df)$ and, as $y \rightarrow x$, $y \in \text{dom}(f)$,

$$\left| f(y) - f(x) - \langle Df(x), y - x \rangle - \frac{1}{2} \langle X_x(y - x), y - x \rangle \right| = o(\|y - x\|^2). \quad (6.4)$$

Proof It is enough to apply Theorem A.2 in [16], which is a refinement of Alexandrov's theorem [20, p. 242]. \square

In the spirit of Proposition 1.1.3 of [9], we prove also:

Corollary 6.2 *Under the same assumptions of Theorem 6.1, let A be defined according to (6.1). Then, for all $x \in A$, there exists $\delta = \delta(x) > 0$ and $c = c(x) \geq 0$ such that for all $v \in \mathbb{R}^n$, $\|v\| = 1$, we have $\frac{\partial^2 f}{\partial v^2} \geq -c$ in the sense of distributions in $B(x, \delta)$, namely for all $\psi \in C_c^\infty(B(x, \delta))$, $\psi \geq 0$, it holds:*

$$\int_{B(x, \delta)} f(x) \frac{\partial^2 \psi}{\partial v^2}(x) dx \geq -c \int_{B(x, \delta)} \psi(x) dx.$$

Furthermore, for a.e. $x \in \Omega$, for all $v \in \mathbb{R}^n$, $\|v\| = 1$, it holds

$$\langle X_x v, v \rangle \geq -2 \max\{\varphi(y, f(y)) : y \in B(x, \delta)\} (1 + \|Df(x)\|^3), \quad (6.5)$$

where X_x is the matrix appearing in the statement of Corollary 6.1.

Proof The first part of the statement is a consequence of the semiconvexity of f on $B(x, \delta)$ (see Proposition 1.1.3 in [9]). To show (6.5), take $v \in \mathbb{R}^n$ such that $\|v\| = 1$ and $x \in A$ such that (6.4) holds. Recalling (1) in Theorem 3.2, we know that f is φ_0 -convex of order 3 in $B(x, \delta)$, with $\varphi_0 = \max\{\varphi(y, f(y)) : y \in \overline{B}(x, \delta)\}$. Then

$$f(x + tv) \geq f(x) + t \langle Df(x), v \rangle - \varphi_0 (1 + \|Df(x)\|^3) t^2.$$

Rearranging the above inequality and using (6.4) we obtain

$$\frac{t^2}{2} \langle X_x v, v \rangle + o(t^2) \geq -\varphi_0 (1 + \|Df(x)\|^3) t^2.$$

Dividing by t^2 and passing to the limit for $t \rightarrow 0$ we prove (6.5). \square

7 Estimates on total variations

The following result shows that (locally bounded) functions with φ_0 -convex epigraph, though not necessarily locally Lipschitz, are actually locally BV in the interior of their domain.

Proposition 7.1 *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous with $\text{epi}(f)$ φ -convex. Set $\Omega' = \text{int dom}(f)$, and assume that $f \in \mathcal{L}_{\text{loc}}^\infty(\Omega')$ or, alternatively, that $n = 1$ and $\varphi \equiv \varphi_0 \in \mathbb{R}^+$. Then $f \in BV_{\text{loc}}(\Omega')$.*

Proof By Claim 3 in the proof of Theorem 5.1 there is no loss of generality in assuming that $\Omega' \neq \emptyset$. Let $U \subset \Omega'$ be open, bounded, and such that $\overline{U} \subset \Omega'$. Recalling Theorem 3.2 (3), or the local boundedness assumption, we have that there exist constants m, M such that

$$m < \inf_U f \leq \sup_U f < M,$$

and $f \in L^1(U)$. By Proposition 4.2, $P(\text{epi}(f), U \times (m, M)) < +\infty$, that is

$$\sup \left\{ \int_{U \times (m, M)} \chi_{\text{epi}(f)}(x, t) \operatorname{div} \psi(x, t) dx dt : \psi \in \mathcal{C}_c^\infty(U \times (m, M)), \|\psi\|_\infty \leq 1 \right\} := c < +\infty.$$

Observe that, for all test functions ψ ,

$$\begin{aligned} & \int_{U \times (m, M)} \chi_{\text{epi}(f)}(x, t) \operatorname{div} \psi(x, t) dx dt \\ &= \int_{\mathbb{R}} \left(\int_U \chi_{[f(x), +\infty)}(t) \operatorname{div} \psi(x, t) dx \right) dt \leq c. \end{aligned} \quad (7.1)$$

Fix $\sigma \in \mathcal{C}_c^\infty(U)$, $\|\sigma\|_\infty \leq 1$. Choose, for all $\bar{t} \in (m, M)$, $\tau_{\bar{t}} \in \mathcal{C}_c^\infty(m, M)$ such that $\|\tau_{\bar{t}}\|_\infty \leq 1$ and $\tau_{\bar{t}}(t) = 1$ in a neighborhood of \bar{t} , and set $\psi_{\sigma, \bar{t}}(x, t) = \sigma(x) \tau_{\bar{t}}(t)$. By (7.1), for a.e. $\bar{t} \in (m, M)$ we have

$$\begin{aligned} & P(\{x \in U : f(x) \leq \bar{t}\}, U) \\ &= \sup \left\{ \int_U \chi_{[f(x), +\infty)}(\bar{t}) \operatorname{div} \psi_{\sigma, \bar{t}}(x, \bar{t}) dx : \sigma \in \mathcal{C}_c^\infty(U), \|\sigma\|_\infty \leq 1 \right\}, \end{aligned}$$

and

$$\int_{\mathbb{R}} P(\{x \in U : f(x) \leq \bar{t}\}, U) d\bar{t} \leq c.$$

Recalling Theorem 1, p. 185 in [20], we see that $f \in BV(U)$. \square

Remark 7.8 The function $f(x) = 1/|x|$ for $x \neq 0$, $= 0$ for $x = 0$ has φ -convex (not φ_0 -convex) epigraph, but $f \notin BV_{\text{loc}}(\mathbb{R})$.

Corollary 7.1 *Let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfy the assumptions of Proposition 7.1. Then the set $\{x \in \text{int dom}(f) : \text{ap lim}_{y \rightarrow x} f(y) \text{ does not exist}\}$ is countably \mathcal{H}^{n-1} -rectifiable.*

Proof See Theorem 3.7.8, p. 173, in [3]. \square

Recalling the concept of approximate differential (see [20, p. 123]) we show:

Corollary 7.2 *Under the assumption of Proposition 7.1, f is approximately differentiable in Ω and $\text{ap } Df = Df = D_w f$ \mathcal{L}^n -a.e. in Ω , where $D_w f$ is the vector of distributional partial derivatives of f in Ω .*

Proof By Theorem 5.1, $\text{ap } Df = Df$ \mathcal{L}^n -a.e. in Ω . By Theorem 4, p. 233 in [20], $\text{ap } Df = D_w f$ \mathcal{L}^n -a.e. \square

Concerning the total variation of Df , the following can be easily seen.

Proposition 7.2 *Let $f : \overline{\Omega} \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfy the assumption of Theorem 5.1. Then for \mathcal{L}^n -a.e. $x \in \Omega$ there exists $\delta = \delta(x) > 0$ such that $Df \in BV(B(x, \delta))$.*

Proof It suffices to take δ such that f is semiconvex on $B(x, \delta)$, and apply Theorem 3 p. 240 in [20]. \square

Remark 7.9 The above result is in a sense optimal, since, for example, the function $f(x) = \text{sign}(x) \frac{1}{2\sqrt{|x|}}$ (which for a.e. x is $D\sqrt{|x|}$) is not $BV_{\text{loc}}(\mathbb{R})$. This marks a difference between convex functions and functions whose epigraph is φ_0 -convex.

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